Off-diagonal online size Ramsey numbers for paths

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Abstract

Consider the following Ramsey game played on the edge set of $K_{\mathbb{N}}$. In every round, Builder selects an edge and Painter colours it red or blue. Builder's goal is to force Painter to create a red copy of a path P_k on k vertices or a blue copy of P_n as soon as possible. The online (size) Ramsey number $\tilde{r}(P_k, P_n)$ is the number of rounds in the game provided Builder and Painter play optimally. We prove that $\tilde{r}(P_k, P_n) \le (5/3 + o(1))n$ provided $k = o(n)$ and $n \to \infty$. We also show that $\tilde{r}(P_4, P_n) \le \lceil 7n/5 \rceil - 1$ for $n \ge 10$, which improves the upper bound obtained by J. Cyman, T. Dzido, J. Lapinskas, and A. Lo and implies their conjecture that $\tilde{r}(P_4, P_n) = \lceil 7n/5 \rceil - 1$.

1 Introduction

Let G and H be finite graphs. Consider the following game $\tilde{R}(G, H)$ played on the infinite board $K_{\mathbb{N}}$ (i.e. the board is a complete graph with the vertex set $\mathbb{N}$). In every round, Builder chooses a previously unselected edge of $K_{\mathbb{N}}$ and Painter colours it red or blue. The game ends when there is a red copy of G or a blue copy of a H on the board. Builder aims to finish the game as soon as possible, while Painter tries to avoid a red G and a blue H as long as possible. By $\tilde{r}(G, H)$ we denote the number of rounds in the game $\tilde{R}(G, H)$, provided both players play optimally and we call it the online size Ramsey number for G and H. In the literature online size Ramsey numbers are called also online Ramsey numbers. The online size Ramsey numbers $\tilde{r}(G, H)$ are game counterparts of well known size Ramsey numbers; the size Ramsey number $\hat{r}(G, H)$ is the minimum number of edges in a graph with the property that every two-colouring of its edges results in a red copy of G or a blue copy of H. Clearly $\tilde{r}(G, H) \le \hat{r}(G, H)$.

In this paper we study online size Ramsey numbers for $G = P_k$ and $H = P_n$, where P_t denotes a path on t vertices. Games in which Builder tries to force Painter to create a monochromatic path were considered also in other variants: the induced version of the online Ramsey number was studied in [3], ordered path games on the infinite (ordered) complete graph and hypergraphs – in [1] and [6]. Studying size Ramsey numbers for paths has much longer history. Let us mention the breakthrough result by Beck [2] that $\hat{r}(P_n, P_n)$ is linear. This result implies that also $\tilde{r}(P_n, P_n)$ is linear, as well as $\tilde{r}(P_k, P_n)$ for a fixed k, since $\tilde{r}(P_k, P_n) \le \hat{r}(P_k, P_n)$ and we have also an easy general bound $\tilde{r}(G, H) \ge |E(G)| + |E(H)| - 1$. However, it seems not easy to find a multiplicative constant c (if exists) such that $\tilde{r}(P_k, P_n) = cn + o(n)$. In general, we have $\tilde{r}(P_k, P_n) \le 2n + 2k - 7$ for every $k, n \ge 2$, proved by Grytczuk, Kierstead and Prałat [5], while the best lower bound for $n \ge k \ge 5$

is $\tilde{r}(P_k, P_n) \geq 3n/2 + k/2 - 7/2$ by Cyman, Dzido, Lapinskas and Lo [4]. Thus we know that $3n/2 + o(n) \leq \tilde{r}(P_k, P_n) \leq 2n + o(n)$ for k fixed and $n \to \infty$. The authors of [4] posed the following conjecture.

Conjecture 1.1 ([4])**.** *For every $k \geq 5$*

$$\frac{\tilde{r}(P_k, P_n)}{n} \to \frac{3}{2} \quad \textit{if } n \to \infty.$$

We make a step towards this conjecture and prove that $\tilde{r}(P_k, P_n) \leq 5n/3 + o(n)$.

Theorem 1.2. *Let $n, k \in \mathbb{N}$ and $k \geq 5$. Then*

$$\tilde{r}(P_k, P_n) \leq \frac{5}{3}n + 12k.$$

We made no effort to optimise the constant 12 in this theorem. Section 3 contains the proof.

There are very few exact results for $\tilde{r}(P_k, P_n)$. It is known that $\tilde{r}(P_3, P_n) = \lceil 5(n-1)/4 \rceil$ for $n \geq 3$ ([4]) and there are computer calculated numbers $\tilde{r}(P_k, P_n)$ for all $k, n \leq 9$ by Prałat [7]. It is proved in [4] that $7n/5 - 1 \leq \tilde{r}(P_4, P_n) \leq 7n/5 + 9$ for $n \geq 4$ and the authors conjectured that the lower bound is tight.

Conjecture 1.3 ([4])**.** *For every $n \geq 4$*

$$\tilde{r}(P_4, P_n) = \left\lceil \frac{7}{5}n \right\rceil - 1.$$

In view of the above mentioned computer calculation, the conjecture is true for $n \leq 9$. We prove that it is true also for $n \geq 10$, by improving the upper bound in [4].

Theorem 1.4. *For every $n \geq 10$*

$$\tilde{r}(P_4, P_n) \leq \left\lceil \frac{7}{5}n \right\rceil - 1.$$

We prove this theorem in Section 5. Our argument is inductive, technically quite complicated, and it is quite different from the argument in [4]. Lately Theorem 1.4 has been proved independently by Yanbo Zhang and Yixin Zhang [8].

2 Preliminaries

For a graph $H = (V, E)$, we put $v(H) = |V|$ and $e(H) = |E|$. By the sum of graphs G and G' we mean the union $G \cup G' = (V(G) \cup V(G'), E(G) \cup E(G'))$.

We say that a graph H is *coloured* if every its edge is blue or red. A graph is red (or blue) if all its edges are red (blue). We assume that also $\emptyset$ is a coloured graph, which is somewhat non-standard. Thus it may happen that a coloured graph or a subgraph of a coloured graph has 0 vertices.

Let H be a coloured graph. By *a component* of H we mean a maximal (in sense of inclusion) connected coloured subgraph contained in H. If $A, B \subseteq V(H)$, then $E_H(A, B)$ denotes the set of all coloured edges of H with one end in A and the other end in B.

Given $n \geq k \geq 2$, and a coloured graph H (it may be empty), consider the following auxiliary game $\mathrm{RR}_H(P_k, P_n)$. The board of the game is $K_{\mathbb{N}}$, with exactly $e(H)$ edges coloured and these edges induce a copy of H in $K_{\mathbb{N}}$. The rules of selecting and colouring edges by Builder and Painter are the same as in the standard game $\tilde{R}(P_k, P_n)$, however, Painter is not allowed to colour an edge red if that would create a red P_k. Builder wins $\mathrm{RR}_H(P_k, P_n)$ at the moment a blue copy of P_n appears on the board. It is not hard to observe that if Builder has a strategy such that $\mathrm{RR}_\emptyset(P_k, P_n)$ ends within at most t rounds, then Builder in $\tilde{R}(P_k, P_n)$ can apply such a strategy as well and finish the game within at most t rounds.

After every round of $\mathrm{RR}_H(P_k, P_n)$, the coloured graph induced by all edges coloured in the game so far (including the edges of H) is called *the host graph.* We say a vertex of $K_\mathbb{N}$ is *free* in a round of the game if it is not a vertex of the host graph.

We say that Builder *forces* an edge uw blue if he selects uw and Painter has to colour it blue according to the rules of the game, i.e. one of u, w, say u, is an end of a red path on $k-1$ vertices, all distinct from w.

3 Proof of Theorem 1.2

Let $n, k \in \mathbb{N}$ and $k \geq 5$. It is enough to show a strategy for Builder in $\mathrm{RR}_\emptyset(P_k, P_n)$ such that the game ends after at most $5n/3 + 12k$ rounds,

In order to simplify the description of Builder's strategy, we assume that Builder can select an edge already coloured. In such a round Painter "colours" it with the same colour the edge already had. Clearly allowing such moves cannot help Builder and may only increase the length of the game.

We divide the game $\mathrm{RR}_\emptyset(P_k, P_n)$ into three stages. Roughly speaking, in the first stage Builder creates many blue paths on 3 vertices. In the second stage he connects them into at most $k-1$ longer paths, while in the last stage he connects a small number of blue paths into a blue path P_n. In order to simplify the description of all three stages, we present a few lemmata.

Lemma 3.1. *Let $n, k \in \mathbb{N}$ and $T = \lceil n/3 \rceil + k$. Builder has a strategy in $RR_\emptyset(P_k, P_n)$ such that after at most $3T + 2(k-1)$ rounds the host graph contains T blue, vertex-disjoint paths of length 2.*

Proof. We will describe the strategy of Builder based on a definition of active and inactive edges. After every round we will call every coloured edge either active or inactive. An active edge may become inactive after a few rounds but an inactive edge stays inactive forever. Given a round of the game, the coloured graph induced by all active edges is called the active graph, while by the inactive graph we mean the coloured graph induced by all inactive edges. Here is the inductive definition of active and inactive edges.

Before the first round there are no active nor inactive edges and the active and inactive graphs are empty sets. Let $t \geq 0$ and suppose that after t rounds we have the active graph A and the inactive graph I. In the next round Builder chooses an end x of a longest red path P in the active graph (if $A = \emptyset$, then x is any free vertex) and selects xy, where y is any free vertex of the board.

Suppose Painter colours xy red. It means that P had less than $k-2$ vertices, otherwise a red P_k would have appeared. Then the red edge xy becomes active. After the $(t+1)$-th round the set of active edges is $E(A) \cup \{xy\}$ and $E(I)$ is the set of inactive edges.

Suppose Painter colours xy blue. Then we have two possibilities. First assume there is a blue edge $e \neq xy$ incident to x. Then the edges xy and e become inactive. If P has a positive length, then the red edge $x'x$, with some $x' \in V(P)$, also becomes inactive. Let $S = \{xy, e, x'x\}$ if P has a positive length; otherwise let $S = \{xy, e\}$. Hence after the $(t+1)$-th round the set of active edges is $E(A) \setminus S$, while $E(I) \cup S$ is the set of inactive edges.

Assume now that there is no blue edge $e \neq xy$ incident to x. Then the edge xy becomes active. After the $(t+1)$-th round the set of active edges is $E(A) \cup \{xy\}$ and $E(I)$ is the set of inactive edges.

Builder continues selecting edges in the above way until the graph induced by all blue inactive edges has at least $n+3k$ vertices, then he stops. Let us verify that such a strategy satisfies the assertion of the lemma.

A routine inductive argument implies that after every round of the game the following holds.

Proposition 3.2. *After every round (until Builder stops the game) the active graph A and the inactive graph I satisfies the following conditions.*

(i) $E(A) \cap E(I) = \emptyset$.

(ii) No inactive blue edge has an endpoint in $V(A)$.

(iii) A is a sum of a red path P on less than k vertices and a blue matching M such that every edge of M has exactly one end in $V(P)$.

(iv) I is the sum of vertex-disjoint blue paths of length 2 and at most m red edges, where m is the number of these paths.

It follows from the above proposition that the game ends at the moment that the inactive graph I contains exactly $\lceil (n+3k)/3 \rceil = \lceil n/3 \rceil + k = T$ blue, vertex-disjoint paths of length 2 and at most T red edges. At this moment the active graph A has less than $2(k-1)$ edges, in view of part (iii) of the proposition. Thus Builder obtains the required T blue paths within less than $3T + 2(k-1)$ rounds. □

The following definition will be useful in two next lemmata. Given $s, d, m \in \mathbb{N}$ with $d \leq s \leq m$, we say that a coloured graph F is *an essential (s,d,m)-graph*, if it contains s vertex-disjoint paths $G_1, G_2, \ldots, G_s$ satisfying the following conditions.

(i) The paths $G_1, G_2, \ldots, G_s \neq \emptyset$ are blue.

(ii) F contains a red path $P = u_1 u_2 \ldots u_d$ on d vertices such that u_i is an end of the path G_i, for $i = 1, 2, \ldots d$.

(iii) $\sum_{i=1}^{s} v(G_i) = m$.

Then we say that F has *the essential red path P* and s *essential blue paths $G_1, \ldots, G_s$*.

Lemma 3.3. *Suppose that $1 \leq d < k \leq s \leq m$ and after some rounds of the game there is an essential (s,d,m)-graph present on the board, its essential blue paths are $G_1, G_2, \ldots, G_s$ and $P = u_1 u_2 \ldots u_d$ is its essential red path P. Assume that in the next round Builder selects the edge $u_d u_{d+1}$, where u_{d+1} is an end of the path G_{d+1}. Then after every response of Painter there is an essential (s', d', m)-graph on the board such that either $d' = d+1$ and $s' = s$, or $s' = s-1$ and $d' \in \{d, d-1\}$.*

Proof. Let us consider two possible situations after colouring $u_d u_{d+1}$ by Painter. If $u_d u_{d+1}$ is red, then $P' = P \cup \{u_d u_{d+1}\}$ is a red path on $d' = d+1$ vertices. It is not hard to verify that the sum of s blue paths G_i and the red path P' is an essential (s', d', m)-graph that satisfies the required conditions.

Suppose that $u_d u_{d+1}$ is blue. Then we define blue paths $G'_i = G_i$ for $i = 1, 2, \ldots, d-1$, $G'_d = G_d \cup G_{d+1} \cup \{u_d u_{d+1}\}$ and $G'_i = G_{i+1}$ for $i = d+1, d+2. \ldots, s-1$. We also define ends u'_i of the paths G'_i such that $u'_i = u_i$ for $i = 1, 2, \ldots, d-1$ and $u'_i = u_{i+1}$ for $i = d+1, d+2. \ldots, s-1$, while u'_d is one of the ends of G'_d. As for the red path, we put $P' = u_1 u_2 \ldots u_{d-1}$ if $d \geq 2$; otherwise $P' = u'_1$. Thus the red path P' has $d = 1$ or $d-1$ vertices, we have $s-1$ vertex-disjoint blue paths G'_i and $\sum_{i=1}^{s} v(G'_i) = \sum_{i=1}^{s} v(G_i)$. Thus the sum of the red path P' and the blue paths $G'_1, \ldots, G'_{s-1}$ satisfies the required conditions. □

Lemma 3.4. *Let $n, k, T \in \mathbb{N}$ and G be a coloured graph containing T blue, vertex-disjoint paths of length 2. Builder has a strategy in $RR_G(P_k, P_n)$ such that after at most $2T - k$ rounds the host graph contains less than k blue, vertex-disjoint paths on $3T$ vertices in total.*

Proof. Let $B \subseteq G$ be the sum of T blue, vertex-disjoint paths of length 2. Then B is a $(T, 1, 3T)$-essential graph, with the essential red path of length 0, consisting of an end of the first path of B. Without loss of generality we assume that $G = B$.

If $T < k$, the Builder in $\mathrm{RR}_B(P_k, P_n)$ achieves his goal without making any move. Otherwise, Builder selects the edges according to Lemma 3.3, as long as the number of essential blue paths is at least k. In view of this lemma, we can have rounds of two kinds: when the length of the essential red path increases (and the number of essential blue paths does not grow) or when the number of essential blue paths decreases (while the length of the essential red path changes by at most 1). A round of the first kind will be called a red-round while a round of the second kind – a blue-round.

We will prove that after $2T - k$ rounds or sooner the assertion of the lemma holds. Assume for a contradiction that at the end of the $(2T - k)$-th round the number of essential blue paths is at least k. It means that there where at most $T - k$ rounds such that the number of essential blue paths decreased. Hence the number of rounds such that the length of the essential red path increased was at least T. Thus the length of the essential red path at the end of the $(2T - k)$-th round is at least $T - (T - k) = k$, which contradicts the rules of $\mathrm{RR}_G(P_k, P_n)$.

Thus after at most $2T - k$ rounds the host graph contains a sum B' of less than k vertex-disjoint, blue paths such that $v(B') = v(B) = 3T$, what follows from Lemma 3.3. □

Next two lemmata, useful in the analysis of the third stage, need a definition of a fence.

Given $s, d, m \in \mathbb{N}$ with $s \leq m$, a coloured graph F is called *an (s, d, m)-fence*, if it contains s vertex-disjoint paths $G_1, G_2, \ldots, G_s$ satisfying the following conditions.

(i) The path $G_1 = w_1 \ldots w_d w_{d+1} \ldots w_j$ has at least one vertex, the path $P = w_1 \ldots w_d$ is red, while the path $G_0 = w_d w_{d+1} \ldots w_j$ is blue.

(ii) Paths $G_2, \ldots, G_s \neq \emptyset$ are blue.

(iii) $v(G_0) + \sum_{i=2}^{s} v(G_i) = m$.

Then we say that F has *the red picket* P and s *blue pickets* $G_0, G_2, \ldots, G_s$.

Lemma 3.5. *Suppose that $m, s \geq 2$, $d \geq 1$, and there is an (s, d, m)-fence on the board. Then Builder can play in such a way that after two rounds the host graph contains an (s', d', m')-fence such that $m' \geq m - 1$ and either $d' \geq d + 1$ and $s' \leq s$, or $s' \leq s - 1$ and $d' \geq d - 1$.*

Proof. Suppose there is an (s, d, m)-fence on the board, with the red picket $P = w_1 \ldots w_d$ and blue pickets $G_0, G_2, \ldots, G_s$. Assume that w_d is an end of G_0, while u_2, u_2' are the ends of G_2. Builder selects the edge $w_d u_2$.

First assume that Painter colours $w_d u_2$ red. Then in the next round Builder selects the same edge $w_d u_2$. After this pair of rounds we define a red path $P' = w_1 \ldots w_d u_2$ and the blue paths $G_0' = G_2$, $G_2' = G_0 \setminus \{w_d\}$, $G_i' = G_i$ for $i = 3, 4, \ldots, s$. The paths $G_1' = G_0' \cup P', G_2', G_3', \ldots, G_s'$ are vertex-disjoint, the red path P' has $d + 1$ vertices, $v(G_0') + \sum_{i=2}^{s} v(G_i') = m - 1$, and among $G_0', G_2', G_3', \ldots, G_s'$ only G_2' may be the empty set. So $\bigcup_{i \leq s \colon G_i' \neq \emptyset} G_i'$ is an $(s', d + 1, m - 1)$-fence with $s' \in \{s, s - 1\}$.

Now assume that Painter colours $w_d u_2$ blue. Then in the next round Builder selects the edge $w_{d-1} u_2'$, provided $d \geq 2$ (if $d = 1$, then Builder selects the same edge $w_d u_2$). Painter colours it red or blue. After thise pair of rounds we define a red path P' in the following way: If $d=1$, then $P'=u_2'$; otherwise if $w_{d-1}u_2'$ is red, then $P'=w_1\dots w_{d-1}u_2'$, while if $w_{d-1}u_2'$ is blue, then $P'=w_1\dots w_{d-1}$. We also define blue paths $G_i'=G_i$ for $i=3,4,\dots,s$ and a path G_0': if $w_{d-1}u_2'$ is red or $d=1$, then $G_0'=G_0\cup G_2\cup\{w_du_2\}$; if $w_{d-1}u_2'$ is blue, then $G_0'=G_0\cup G_2\cup\{w_du_2,w_{d-1}u_2'\}$. Notice that the paths $G_1'=G_0'\cup P',G_3',G_4',\dots,G_s'$ are vertex-disjoint, the red path P' has d or $d-1$ vertices, $v(G_0')+\sum_{i=3}^s v(G_i')\in\{m,m+1\}$, and every path $G_0',G_2',G_3',\dots,G_s'$ has at least one vertex. So $G_1'+\bigcup_{i=3}^s G_i'$ is an $(s-1,d',m')$-fence with $d'\le d$ and $m'\ge m$.

In both cases the assertion follows. □

Lemma 3.6. *Let $n,k,t\in\mathbb{N}$ and G' be a coloured graph containing a sum B of t blue, vertex-disjoint paths. Then Builder has a strategy in $RR_{G'}(P_k,P_n)$ such that after at most $4t+2k$ rounds the host graph contains a blue path on at least $v(B)-2t-k$ vertices.*

Proof. Without loss of generality we assume that $G'=B$. Observe than B consisting of t blue, vertex-disjoint paths is a $(t,1,v(B))$-fence, provided we define the red picket of length 0 to consist of an end of the first path of B.

Builder in $\mathrm{RR}_B(P_k,P_n)$ selects the edges according to Lemma 3.5, as long as the number of blue pickets is at least 2. In view of this lemma, we have pairs of rounds of two kinds: when the length of the red picket increases (and the number of blue pickets does not grow) or when the number of blue pickets decreases (while the length of the red picket decreases by at most 1). A pair of rounds of the first kind will be called a red pair of rounds while a pair of rounds of the second kind – a blue-pair of rounds. After every pair of rounds (red or blue) the sum of the number of vertices in blue pickets decreases by at most one.

We will prove that after at most $2t+k$ pairs of rounds Builder achieves his goal. Assume for a contradiction that at the end of $(2t+k)$-th pair of rounds there are at least two blue pickets. It means that there where at most $t-2$ blue-pairs of rounds. Hence the number of red-pairs of rounds was greater than $t+k$. Thus the length of the red picket is greater than $(t+k)-(t-2)>k$, which contradicts the rules of $\mathrm{RR}_{G'}(P_k,P_n)$.

Thus after at most $2(2t+k)=4t+2k$ rounds the host graph contains only one blue picket and, in view of Lemma 3.5, the number of its vertices is at least $v(B)-(2t+k)$. □

We are ready to prove the main theorem.

Proof of Theorem 1.2. We present Builder's strategy in the game $\mathrm{RR}_\emptyset(P_k,P_n)$ in three stages. Let $T=\lceil n/3\rceil+k$.

Stage 1.

Builder plays according to a strategy whose existence is guaranteed by Lemma 3.1 and within at most $3T+2(k-1)$ rounds he obtains a coloured graph G containing T blue, vertex-disjoint paths of length 2. The game proceeds to the second stage, equivalent to the game $\mathrm{RR}_G(P_k,P_n)$.

Stage 2.

In this stage Builder selects the edges according to Lemma 3.4 and within at most $2T-k$ rounds of Stage 2 he obtains a coloured graph G' containing a sum B of $t<k$ blue, vertex-disjoint paths, such that $v(B)=3T$. The game proceeds to the third stage, equivalent to the game $\mathrm{RR}_{G'}(P_k,P_n)$.

Stage 3.

In the last stage Builder applies a strategy from Lemma 3.6 and after at most $4t+2k$ rounds of Stage 3 the host graph contains a blue path P on at least $v(B)-2t-k$ vertices. Then Stage 3 ends.

Let us analyse the host graph after all three stages of the game. For the blue path P obtained at the end of Stage 3 we have

$$v(P) \geq v(B) - 2t - k > v(B) - 3k = 3T - 3k \geq n + 3k - 3k = n.$$

The number of rounds in all stages is not greater than

$$(3T + 2(k-1)) + (2T - k) + (4t + 2k) \leq 5T + 7k - 6 < 5\Big(\frac{n}{3} + k\Big) + 7k = \frac{5}{3}n + 12k.$$

So a blue path on n vertices was created within at most $5n/3 + 12k$ rounds and the proof of Theorem 1.2 is complete. □

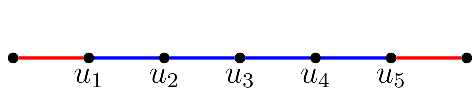
4 Tools for studying the P_4 versus P_n game

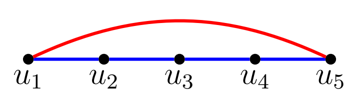
Before the proof of Theorem 1.4 we need a few additional definitions and lemmata.

If G and G' are coloured graphs and $G' \subseteq G$, then we denote by $\mathrm{Red}_G(G')$ and $\mathrm{Blue}_G(G')$ the sets of all red edges and blue edges of G respectively, with at least one end in $V(G')$. After every round of a game $\mathrm{RR}_H(P_4, P_n)$, if G is the host graph, for every coloured graph $G' \subseteq G$ we define $\mathrm{Red}(G') = \mathrm{Red}_G(G')$ and $\mathrm{Blue}(G') = \mathrm{Blue}_G(G')$.

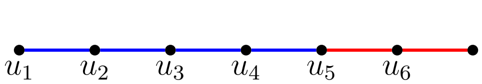
Let $c_1, c_2, \ldots, c_k \in \{b, r\}$ be consecutive edge colours of a coloured path P_{k+1}. Then the coloured path is called a $c_1c_2\ldots c_k$-path. Suppose $P \subseteq K_{\mathbb{N}}$ is a blue path with ends x_1, x_2, and there is a red path of length $l_i \geq 0$ with an end x_i, for $i = 1, 2$. Then P is called a blue (l_1, l_2)-path and x_i is called its l_i-end. The coloured path $u_1u_2\ldots u_ku_{k+1} \subseteq K_{\mathbb{N}}$ such that $k \geq 2$, the edge u_ku_{k+1} is red and $u_1u_2\ldots u_k$ is a blue $(2,0)$-path with a 2-end u_k, will be called *an extended* $(2,0)$*-path* with *the blue end* u_1, *the transition vertex* u_k and *the red end* u_{k+1}.

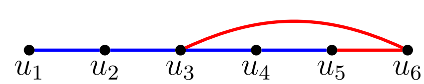
Here are two examples of coloured graphs containing a $(1,1)$-path P with $V(P) = \{u_1, u_2, u_3, u_4, u_5\}$ and 1-ends u_1, u_5:



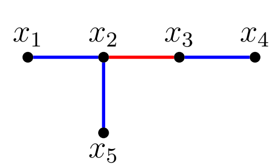


Below there are two examples of coloured graphs containing an extended $(2,0)$-path P on the vertex set $V(P) = \{u_1, u_2, u_3, u_4, u_5, u_6\}$, with the blue end u_1, the red end u_6 and the transition vertex u_5:





The following coloured graph with one red and three blue edges will be called *a limb*.



Suppose that H is a coloured graph that contains vertex-disjoint coloured subgraphs G_0, G_1, G_2, L satisfying the following conditions.

(A) Either $L = \emptyset$, or L is a limb and it is a component of H.

(B) Either $V(G_0) = \emptyset$, or G_0 is a blue path on at least one vertex such that $G_0 \cup \mathrm{Red}_H(G_0)$ is a component of G. Furthermore $|\mathrm{Red}_H(G_0)| \leq \lceil \frac{2}{5} v(G_0) \rceil - 1$.

(C) $G_2 = \emptyset$ or G_2 is an extended $(2,0)$-path on at least three vertices. Either $G_1 = \emptyset$, or G_1 is a blue $(1,1)$-path on at least one vertex such that neither of its 1-ends is adjacent in G to any of: the blue end of G_2, the transition vertex of G_2, the red end of G_2. Furthermore $\mathrm{Blue}_H(G_1 \cup G_2) \setminus E(G_1 \cup G_2) = \emptyset$ and $|\mathrm{Red}_H(G_1 \cup G_2)| \leq \lceil \frac{2}{5} v(G_1 \cup G_2) \rceil$.

(D) The set of all blue edges of H is equal to $\mathrm{Blue}_H(G_0 \cup G_1 \cup G_2 \cup L)$.

(E) The set of all red edges of H is equal to $\mathrm{Red}_H(G_0 \cup G_1 \cup G_2 \cup L)$.

(F) $5 \mid v(G_0)$ or $5 \mid v(G_1 \cup G_2)$.

Then we call H *a good graph* with *essential subgraphs* (G_0, G_1, G_2, L). We say that $V(L) \cup V(G_0) \cup V(G_1) \cup V(G_2)$ is *the set of essential vertices* of H and denote the number of essential vertices of H by $\mathrm{ess}(H)$. A good graph is *very good* if it satisfies the additional condition

(G) $5 \mid v(G_0)$ and $5 \mid v(G_1 \cup G_2)$.

By *a simple good graph* we mean a good graph such that at least three of its essential subgraphs are empty sets. Similarly we define *a simple very good graph* H, if H is a very good graph.

Below we present some observations, following immediately from the definition of a good graph, which will be often used, sometimes implicitly, in the paper.

Proposition 4.1. *Suppose that $k \in \mathbb{N}$ and H is a good graph with essential subgraphs (G_0, G_1, G_2, L). Then the following holds.*

(i) The number of red edges of H is not greater than $\lceil \frac{2}{5} \mathrm{ess}(H) \rceil$ and, if $G_0 \neq \emptyset$, not greater than $\lceil \frac{2}{5} \mathrm{ess}(H) \rceil - 1$. The number of blue edges of H is not greater than $\mathrm{ess}(H) - 1$.

(ii) $e(H) \leq \lceil \frac{7}{5} \mathrm{ess}(H) \rceil - 1$.

(iii) $H \setminus G_0$ is a good graph, also $H \setminus L$ is good. If H is very good, then also $H \setminus G_0$ and $H \setminus L$ are very good.

(iv) If $5 \mid v(H)$, then H is very good.

(v) If H' is a sum of a blue path P on $k \geq 1$ vertices and at most $\lceil \frac{2}{5} k \rceil - 1$ red edges incident to the path, then H' is a simple good graph, with k essential vertices and essential subgraphs $(P, \emptyset, \emptyset, \emptyset)$. Furthermore, if $5 \mid k$, then H' is simple very good.

(vi) If H' is a sum of a blue $(1,1)$-path P on $k \geq 1$ vertices and at most $\lceil \frac{2}{5} k \rceil$ red edges incident to the path, then H' is a simple good graph, with k essential vertices and essential subgraphs $(\emptyset, P, \emptyset, \emptyset)$. Furthermore, if $5 \mid k$, then H' is simple very good.

(vii) If H' is a sum of an extended $(2,0)$-path P on $k \geq 3$ vertices and at most $\lceil \frac{2}{5} k \rceil$ red edges incident to the path, then H' is a simple good graph, with k essential vertices and essential subgraphs $(\emptyset, \emptyset, P, \emptyset)$. Furthermore, if $5 \mid k$, then H' is simple very good.

We omit the uncomplicated justification of Proposition 4.1. The following lemma says, among other things, that two disjoint very good graphs can be efficiently connected by Builder, if one of them is simple and different from a limb.

Lemma 4.2. *Suppose that H is a very good graph and H' is a simple good graph with its essential subgraph different from a limb. Then Builder has a strategy in $RR_{H\cup H'}(P_4, P_n)$ such that after a finite number of rounds the graph G induced by all coloured edges of the board is a good graph with $ess(H) + ess(H')$ vertices.*

Furthermore, if H' is a simple very good graph, then the resulting graph G is very good.

Proof. Let (G_0, G_1, G_2, L) be essential subgraphs of H. Since H is very good, we have $5 \mid v(G_0)$ and $5 \mid v(G_1 \cup G_2)$.

We split the argument into a few parts, depending on the essential subgraph of H'. If $H' = \emptyset$, the assertion trivially holds so let $H' \neq \emptyset$.

Case 1. H' has the essential subgraphs $(\emptyset, P, \emptyset, \emptyset)$.

Thus P is a blue path $(1,1)$-path on at least 1 vertex and, in view of Condition (C), we have $|\mathrm{Red}_{H'}(P)| \leq \lceil 2v(P)/5 \rceil$. If $G_1 = \emptyset$ then, since $5 \mid v(G_0)$, clearly $H \cup H'$ is a good graph with essential subgraphs (G_0, P, G_2, L). Thus the required good graph is obtained without any move in the game.

Assume further that $G_1 \neq \emptyset$. Let x, y be the 1-ends of P and u_1, u_1' be the 1-ends of G_1. Builder in $\mathrm{RR}_{H\cup H'}(P_4, P_n)$ forces the edge xu_1 blue. Then a blue $(1,1)$-path G_1' with 1-ends u_1', y appears on the board and it has $v(G_1) + v(P)$ vertices. Let G be the host graph at this moment. Since $5 \mid v(G_1 \cup G_2)$, we have

$$\begin{aligned} |\mathrm{Red}_G(G_1' \cup G_2)| &= |\mathrm{Red}_H(G_1 \cup G_2)| + |\mathrm{Red}_{H'}(P)| \leq \frac{2}{5}v(G_1 \cup G_2) + \left\lceil \frac{2}{5}v(P) \right\rceil \\ &= \left\lceil \frac{2}{5}\big(v(G_1) + v(P) + v(G_2)\big) \right\rceil = \left\lceil \frac{2}{5}v(G_1' \cup G_2) \right\rceil. \end{aligned}$$

Furthermore $\mathrm{Red}_G(G_0) = \mathrm{Red}_H(G_0)$, $\mathrm{Blue}_G(G_0) = \mathrm{Blue}_H(G_0)$, $\mathrm{Red}_G(L) = \mathrm{Red}_H(L)$, $\mathrm{Blue}_G(L) = \mathrm{Blue}_H(L)$, and $5 \mid v(G_0)$ by the assumption that H is very good, so in view of the above estimation of $|\mathrm{Red}_G(G_1' \cup G_2)|$, we conclude that G is a good graph with essential subgraphs (G_0, G_1', G_2, L) and $\mathrm{ess}(G) = \mathrm{ess}(H) + \mathrm{ess}(H')$.

Case 2. H' has the essential subgraphs $(\emptyset, \emptyset, P, \emptyset)$.

Thus P is an extended $(2,0)$-path on at least 3 vertices, with $|\mathrm{Red}_H(P)| \leq \lceil 2v(P)/5 \rceil$. If $G_2 = \emptyset$, then $H \cup H'$ is the required good graph, with essential subgraphs (G_0, G_2, P, L). Assume further that $G_2 \neq \emptyset$.

Let y, x, z be the blue end, the red end and the transition vertex of P, respectively. Similarly, let u_2', u_2, w_2 be the blue end, the red end and the transition vertex of G_2.

Builder in $\mathrm{RR}_{H\cup H'}(P_4, P_n)$ forces the edge zu_2' blue. Then we obtain an extended $(2,0)$-path G_2' on the vertex set $V(G_2 \cup (P \setminus \{x\})$, with its blue end y and its red end u_2, and the blue $(1,1)$-path on one vertex x.

If $G_1 = \emptyset$, we define $G_1' = x$; otherwise let u_1, u_1' be the 1-ends of G_1. In the latter case, in the next round Builder forces the edge xu_1 blue and we define $G_1' = G_1 \cup \{xu_1\}$. In both cases G_1' is a blue $(1,1)$-path on the vertex set $V(G_1) \cup \{x\}$. Let us estimate the number of red edges incident to vertices of $G_1' \cup G_2'$ at this moment of the game, bearing in mind the assumption $5 \mid v(G_1 \cup G_2)$. We have

$$\begin{aligned}|\mathrm{Red}(G_1' \cup G_2')| &= |\mathrm{Red}_{H'}(P)| + |\mathrm{Red}_H(G_1 \cup G_2)| \le \left\lceil \frac{2}{5} v(P) \right\rceil + \frac{2}{5} v(G_1 \cup G_2) \\ &= \left\lceil \frac{2}{5} v(P \cup G_1 \cup G_2) \right\rceil = \left\lceil \frac{2}{5} v(G_1' \cup G_2') \right\rceil.\end{aligned}$$

As in the previous case, adding a blue edge xu_1 to the host graph does not change the set of edges incident to any vertex of G_0 and L, and we have $5 \mid v(G_0)$, thus the obtained host graph is a good graph with essential subgraphs (G_0, G_1', G_2', L), with $\mathrm{ess}(H) + \mathrm{ess}(H')$ essential vertices.

Case 3. H' has the essential subgraphs $(P, \emptyset, \emptyset, \emptyset)$.

Thus P is a blue path on at least 1 vertex and $|\mathrm{Red}_{H'}(P)| \le \lceil 2v(P)/5 \rceil - 1$. If $G_0 = \emptyset$ then, since $5 \mid v(G_1 \cup G_2)$, clearly $H \cup H'$ is a good graph with essential subgraphs (P, G_1, G_2, L). Thus the required good graph is obtained at once.

Assume further that $G_0 \ne \emptyset$, u_0, u_0' are the ends of G_0 and x, y are the ends of P. Builder starts $\mathrm{RR}_{H \cup H'}(P_4, P_n)$ with selecting the edge xu_0.

If Painter colours xu_0 blue, then we obtain a blue path G_0' on $v(G_0) + v(P)$ vertices and

$$\begin{aligned}|\mathrm{Red}(G_0')| &= |\mathrm{Red}_H(G_0)| + |\mathrm{Red}_{H'}(P)| \le \frac{2}{5} v(G_0) - 1 + \left\lceil \frac{2}{5} v(P) \right\rceil - 1 \\ &= \left\lceil \frac{2}{5} (v(G_0) + v(P)) \right\rceil - 2 < \left\lceil \frac{2}{5} v(G_0') \right\rceil - 1.\end{aligned}$$

The above estimation and the fact that adding the blue edge xu_0 does not affect the set of edges incident to G_1, G_2 or L, implies that for $5 \mid v(G_1 \cup G_2)$, the obtained host graph is a good graph with essential subgraphs (G_0', G_1, G_2, L) and $v(G_0') + v(G_1 \cup G_2) + v(L) = \mathrm{ess}(H) + \mathrm{ess}(H')$ essential vertices.

Now assume that Painter colours xu_0 red. Then Builder selects yu_0'. If Painter colours it blue, then a blue $(1,1)$-path G_1' with 1-ends u_0, x appears on the board. Suppose Painter colours yu_0' red. Then, if P has more than one vertex, Builder forces the edge yu_0 blue, and thereby obtains a blue $(1,1)$-path P' with 1-ends u_0', x. If P has only one vertex, we get an extended $(2,0)$-path P', with its blue end u_0' and its red end x, containing the red edge u_0x. In all cases the obtained path P' has $v(G_0) + v(P)$ vertices and, since at most two new red edges (incident to vertices of P') were selected in the game, we have

$$\begin{aligned}|\mathrm{Red}(P')| &\le |\mathrm{Red}_H(G_0)| + |\mathrm{Red}_{H'}(P)| + 2 \le \frac{2}{5} v(G_0) - 1 + \left\lceil \frac{2}{5} v(P) \right\rceil - 1 + 2 \\ &= \left\lceil \frac{2}{5} (v(G_0) + v(P)) \right\rceil = \left\lceil \frac{2}{5} v(P') \right\rceil.\end{aligned}$$

Let H'' be the sum of P' and all (at most two) red edges selected in the game. Then the above estimation and Proposition 4.1(vi),(vii) imply that H'' is a simple good graph. Furthermore, in view of Proposition 4.1(iii), the graph $H \setminus G_0$ is very good. Let F be the host graph at this moment of the game. Observe that F is the sum of $(H \setminus G_0)$ and H'' and all assumptions of Lemma 4.2 are fulfilled by the graphs $H \setminus G_0$ and H''. We have already proved Lemma 4.2 if the simple good graph contains an extended $(2,0)$-path (Case 2) or a blue $(1,1)$-path (Case 1) so we argue that after a few rounds of $\mathrm{RR}_F(P_4, P_n)$ the obtained host graph is a good graph such that its number of essential vertices is

$$\mathrm{ess}(H \setminus G_0) + \mathrm{ess}(H'') = v(G_1 \cup G_2) + v(L) + v(P') = v(G_1 \cup G_2) + v(L) + v(G_0) + v(P) = \mathrm{ess}(H) + \mathrm{ess}(H').$$

Hence the first part of the assertion follows.

The second part of the assertion is a consequence of Proposition 4.1(iv) and the fact that if both graphs H and H' are very good, then $5 \mid \mathrm{ess}(H) + \mathrm{ess}(H')$. $\square$

The next lemma is crucial to the inductive argument in the proof of Theorem 1.4.

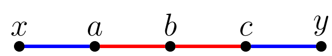
Lemma 4.3. *Suppose that $n \geq 10$, $5 \leq k \leq n$, $5 \mid k$, and H is a coloured graph. Assume that H is good and has $k-5$ essential vertices. Then Builder has a strategy in $RR_H(P_4, P_n)$ such that after a finite number of rounds the host graph is a very good graph with k essential vertices.*

Proof. Suppose that H satisfies the assumption of the lemma and has essential subgraphs (G_0, G_1, G_2, L). Note that H is very good, in view of 4.1(iv).

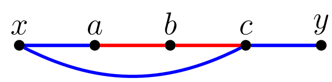
Suppose that $K_{\mathbb{N}}$ contains the coloured graph H and $K'_{\mathbb{N}} \subseteq K_{\mathbb{N}}$ is the complete graph, vertex-disjoint from H. Builder starts $\mathrm{RR}_H(P_4, P_n)$ by selecting two adjacent edges of $K'_{\mathbb{N}}$, say ab and bc. We consider all Painter's responses.

Case 1. ab and bc are red.

Builder selects edges ax and cy for any free, distinct $x, y \in V(K'_{\mathbb{N}})$. According to the rules of the game $\mathrm{RR}_H(P_4, P_n)$, Painter has to colour them blue so the $brrb$-path is obtained:



Then Builder forces xc blue and the following coloured graph appears:



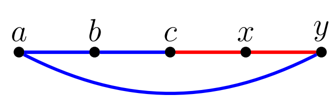
Thus it contains an extended $(2, 0)$-path $ycxab$. In view of Proposition 4.1(vii), the coloured graph H' induced by the five edges selected in the game is a simple very good graph with five essential vertices. Since the host graph at this moment is the sum of vertex-disjoint very good graphs H and H', we can apply Lemma 4.2 and we infer that Builder can continue the game so that after some rounds the host graph is a very good graph, with $\mathrm{ess}(H) + \mathrm{ess}(H') = k$ essential vertices.

Case 2. ab and bc are blue.

In the next two rounds Builder selects edges cx and xy for any free, distinct $x, y \in V(K'_{\mathbb{N}})$. Let us consider four cases, depending on Painter's response.

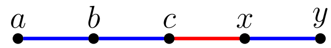
If Painter colours them blue, we obtain a blue path $H' = abcxy$ such that no red edge is incident to it. Based on Proposition 4.1(v), it is a simple very good graph on five vertices and again we apply Lemma 4.2 to H and H'. Thereby we infer that after some rounds Builder obtains a very good host graph with $\mathrm{ess}(H) + \mathrm{ess}(H') = k$ essential vertices.

If Painter colours cx and xy red, then Builder forces the edge ay blue and we obtain an extended $(2, 0)$-path $cbayx$.

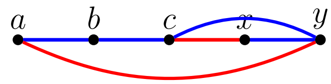


The coloured graph H' induced by the five edges selected in the game is a simple very good graph with five essential vertices, we apply Lemma 4.2 again, and conclude that after some rounds Builder obtains a very good graph with $\mathrm{ess}(H) + \mathrm{ess}(H') = k'$ essential vertices.

Suppose Painter colours cx red and xy blue. So we obtain a $bbrb$-path:



Then Builder selects ay. If Painter colours it blue, we receive a blue $(1,1)$-path on five vertices, with 1-ends x, c. If Painter colours ay red, Builder forces cy blue and we receive the following coloured graph.



It is a blue $(1,1)$-path with 1-ends x, a.

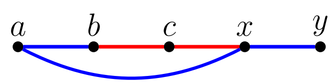
Thus, regardless of whether Painter coloured ay red or blue, Builder obtains a graph H' which is a sum of a blue $(1,1)$-path on five vertices and at most two red edges incident to the path. As before, we apply Proposition 4.1(vi), then Lemma 4.2 to H' and H, and conclude the assertion.

Suppose Painter colours cx blue and xy red. Then Builder selects ay. If Painter colours it blue, we receive a blue $(1,1)$-path on five vertices, with 1-ends x, y. If Painter colours ay red, we receive an extended $(2,0)$-path $abcxy$. The analysis is analogous as before and we conclude the assertion in a similar way.

Case 3. ab and bc have different colours, i.e. ab is, say, blue and bc is red.

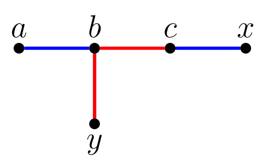
Suppose first that $L = \emptyset$. Builder selects cx for any free $x \in V(K'_{\mathbb{N}})$.

If Painter colours it red, then Builder forces an edge xy blue for any free $y \in V(K'_{\mathbb{N}})$, then he forces ax blue. The following coloured graph is obtained:

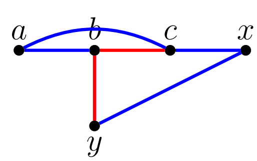


Such a graph with an extended $(2,0)$-path has been already analysed in Case 1.

Suppose Painter colours cx blue. Then Builder selects by for any free $y \in V(K'_{\mathbb{N}})$. If Painter colours it blue, the four edges selected in the game so far form a limb L', vertex-disjoint with H. Hence, since H is very good and $L = \emptyset$, also $H \cup L'$ is a very good graph. It has $\mathrm{ess}(H) + 5 = k$ essential vertices and the assertion follows. Assume that Painter colours by red. We receive the following coloured graph at $K'_{\mathbb{N}}$:



Then Builder forces edges ac and xy blue and we obtain a graph H' which is a sum of a blue $(2,1)$-path (which is also a $(1,1)$-path) $yxcab$ and a red path of length 2, as we illustrate in the following picture.

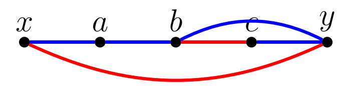


Such a graph is simple very good so based on Lemma 4.2 applied to H' and H, after some rounds Builder obtains a very good graph with $\mathrm{ess}(H) + \mathrm{ess}(H') = k$ essential vertices.

Now suppose that $L \neq \emptyset$. Then Builder selects an edge xy for any free $x, y \in V(K'_{\mathbb{N}})$.

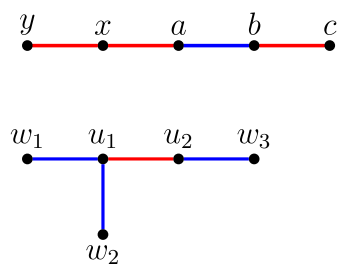
If Painter colours xy blue, then Builder selects cx. No matter how Painter colours it, the four edges selected in the game form either a $brrb$-path considered in Case 1 or a $bbrb$-path, which we considered in Case 2. Therefore further we assume that Painter colours xy red. Then Builder selects ax.

If Painter colours ax blue, then Builder forces by and cy blue. The following graph is obtained:

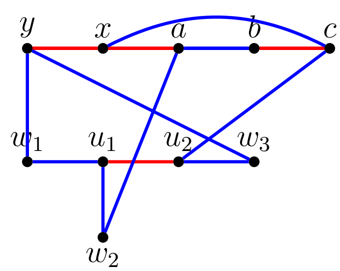


Such a stage of the game has been already analysed in Case 2.

Further we assume that ax is red. Consider two components of the coloured graph induced by all coloured edges present on the board $K_{\mathbb{N}}$: the $rrbr$-path created in the four rounds of the game and the limb L, with its vertices denoted as in the following picture.



In the next five rounds Builder forces blue edges: yw_1, yw_3, cu_2, cx, aw_2. After we obtain the following coloured graph H':



Thus H' is a sum of a blue $(1,1)$-path on 10 vertices and four red edges. We know by Proposition 4.1(vi) that such a graph is simple very good, and that, by Proposition 4.1(iii), the graph $H \setminus L$ is very good. The very good graphs H' and $H \setminus L$ are vertex-disjoint and they satisfy the assumption of Lemma 4.2. Similarly to the previous analysis, based on Lemma 4.2 Builder obtains a very good graph with $\mathrm{ess}(H \setminus L) + \mathrm{ess}(H') = k$ essential vertices. $\square$

An immediate consequence of Lemma 4.3 and the inductive argument is the following corollary.

Corollary 4.4. *Suppose that $n \geq 10$. Then Builder has a strategy in $RR_{\emptyset}(P_4, P_n)$ such that after a finite number of rounds the host graph is a very good graph with $5\lfloor n/5 \rfloor$ essential vertices.*

So far Lemma 4.2 was applied to very good graphs only. In the next section we will use it also in case of a simple good graph H' which is not very good, i.e. with the number of vertices not divisible by 5.

5 Proof of Theorem 1.4

Let $n = m + r$ with $5 \mid m$, $m \geq 10$ and $0 \leq r \leq 4$. It is enough to show a strategy for Builder in $\mathrm{RR}_{\emptyset}(P_4, P_n)$ such that the game ends after at most $\lceil 7n/5 \rceil - 1$ rounds. We divide the game $\mathrm{RR}_{\emptyset}(P_4, P_n)$ into three stages. Roughly speaking, in the first stage Builder creates a big very good graph and in the second stage he increases it to a good graph with n essential vertices. In the last stage he connects parts of the good graph into a blue path P_n.

Stage 1.

In the first stage, based on Corollary 4.4, Builder uses a strategy which guarantees that after some round the host graph is a very good graph H with m essential vertices. Then the first stage ends. Assume that the very good graph H has essential subgraphs (G_0, G_1, G_2, L). The game proceeds to the next stage, equivalent to the game $\mathrm{RR}_H(P_4, P_n)$.

Stage 2.

Let $K'_{\mathbb{N}} \subseteq K_{\mathbb{N}}$ be a complete graph, vertex-disjoint from H. Builder begins by selecting $r-1$ edges of a path on r vertices in $K'_{\mathbb{N}}$ (if $r \leq 1$, he does nothing). After Painter colours them, we obtain a coloured path P on r vertices. For $r = 0$ we put $P = \emptyset$, for $r = 1$ we have a trivial path P. We consider a few cases in order to define a new coloured component H'. The only case not listed below is when $r = 2$ and P is a red edge. We call it *the exceptional case*, assume that the second stage ended here and consider this case later.

Case 1. $0 \leq r \leq 4$ and P is blue.

Then we put $H' = P$. There is no red edges incident to the path H' so it is a simple good graph with $\mathrm{ess}(H') = r$.

Case 2. $r = 3$, $P = abc$ and it is an rr-path.

Then Builder forces blue edges ax, xc with any free $x \in V(K'_{\mathbb{N}})$. Hence we obtain a coloured graph H' that is the sum of a blue $(1,1)$-path $G_1 = axc$ and two red edges. H' is a simple good graph since $|\mathrm{Red}(H')| = 2 = \lceil 2v(G_1)/5 \rceil$. Clearly $\mathrm{ess}(H') = r$.

Case 3. $r = 3$, $P = abc$ and it is a br-path.

Then Builder selects ac. No matter how Painter colours it, the coloured graph induced by the three edges form a simple good graph H' with 3 essential vertices. Indeed, if ac is blue, then H' contains a blue $(1,1)$-path cab; if ac is red, then H' contains an extended $(2,0)$-path abc.

Case 4. $r = 4$, $P = abcd$ and at least one edge of P is red.

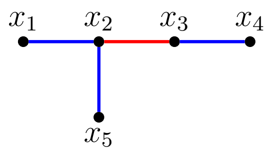
Then Builder selects ad. Thus the four selected edges form a cycle. If three edges of the cycle are blue, we get a blue $(1,1)$-path on 4 vertices. If two edges of the cycle are blue and two red edges are adjacent, we get an extended $(2,0)$-path on 4 vertices. If the two red edges are disjoint, say ab and cd, then Builder forces a blue edge ac and we obtain a blue $(1,1)$-path on 4 vertices. Obviously, the cycle cannot have exactly one edge blue, since Painter never creates a red P_4. Thus in all cases for $r = 4$ the four coloured edges form a simple good graph and we denote it by H'. We have $\mathrm{ess}(H') = 4$.

It follows from the argument above that for every $r \leq 4$, in every case apart from the exceptional case ($r = 2$ and P red) the new component H' is a simple good graph with r essential vertices. Furthermore, H and H' satisfy the assumption of Lemma 4.2. Thus within some further rounds Builder creates a good graph G with $\mathrm{ess}(G) = m+r = n$ and with essential subgraphs (G'_0, G'_1, G'_2, L).

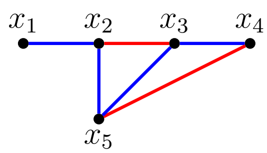
The second stage ends and the game proceeds to the third stage, equivalent to the game $\mathrm{RR}_G(P_4, P_n)$.

Stage 3.

Let us recall that we still exclude the exceptional case from the analysis. Builder begins by transforming the limb L (if $L \neq \emptyset$) into a blue $(1,0)$-path. Suppose the vertices of L are denoted as in the following picture.



Builder selects x_4x_5. If Painter colours it blue, we obtain a blue $(1,0)$-path $P' = x_3x_4x_5x_2x_1$. If Painter colours x_4x_5 red, Builder forces the edge x_5x_3 blue and we get a blue $(1,0)$-path $P' = x_1x_2x_5x_3x_4$, as shown in the following picture.



Without loss of generality we can assume that x_4 is an 1-end of the path P'. Clearly $|\mathrm{Red}(P')| \leq 2 = 2v(P')/5$. If $G_1' = \emptyset$, we put $F_1 = P'$; otherwise for an 1-end u_1 of G_1, Builder forces the edge x_4u_1 blue. Then we define $F_1 = P' \cup G_1' \cup \{x_4u_1\}$.

For $L = \emptyset$ we define $F_1 = G_1'$. Thus in both cases (L empty or not) F_1 is a $(1,0)$-path or $F_1 = \emptyset$. Observe also that in both cases at this moment of the game we have $v(F_1) = v(L) + v(G_1')$ and $|\mathrm{Red}(L)| = 2v(L)/5$.

If $F_1 = \emptyset$ or $G_2' = \emptyset$, we define $D = F_1 \cup G_2'$. Suppose now that $F_1, G_2' \neq \emptyset$. Let f_1 be an 1-end of the path F_1, f_1' be its other 1-end, and let u_2', w_2, u_2 denote the blue end, the transition vertex and the red end of G_2', respectively. Breaker forces the edges f_1u_2 and w_2f_1' blue. After there is a blue $(1,0)$-path D on $v(F_1) + v(G_2')$ vertices on the board, with an 1-end u_2 and the other end u_2'.

In both cases ($F_1, G_2' \neq \emptyset$ or not) we have $v(D) = v(F_1) + v(G_2') = v(L) + v(G_1') + v(G_2')$ and every red edge incident to a vertex of D is either a red edge of G incident to G_1' or G_2', or it is incident to L. Thus

$$|\mathrm{Red}(D)| = |\mathrm{Red}_G(G_2' \cup G_1')| + |\mathrm{Red}(L)| = |\mathrm{Red}_G(G_2' \cup G_1')| + \frac{2}{5}v(L). \tag{1}$$

Furthermore, every blue edge of $L \cup G_1' \cup G_2'$ is a blue edge of D.

It remains to connect D and G_0' into a blue path D'. Let x be an 1-end of D and x' be the other end of it. If $G_0' = \emptyset$, then we define $D' = D$. Clearly then $|\mathrm{Red}(D')| = |\mathrm{Red}(D)|$ and

$$\begin{aligned}
|\mathrm{Red}(D')| &\leq |\mathrm{Red}_G(G_2' \cup G_1')| + \frac{2}{5}v(L) \leq \left\lceil \frac{2}{5}v(G_1' \cup G_2') \right\rceil + \frac{2}{5}v(L) \\
&= \left\lceil \frac{2}{5}(\mathrm{ess}(G) - v(L)) \right\rceil + \frac{2}{5}v(L) = \left\lceil \frac{2}{5}\mathrm{ess}(G) \right\rceil.
\end{aligned}$$

Suppose that $G_0' \neq \emptyset$. Let u_0 and u_0' be the ends of the path G_0'. Builder selects the edge xu_0. If Painter colours it blue, then we define $D' = D \cup G_0' \cup \{xu_0\}$. Suppose Painter colours xu_0 red. Then u_0 becomes a 2-end of G_0'. We also know that there is a red edge, say e, incident to x but different from xu_0. If e is not incident to x', then Builder forces the edge u_0x' blue. If e is incident to x', then x' is a 2-end of D now and Builder can force blue either the edge $x'u_0'$ if $u_0' \neq u_0$, or an edge $x'y$ with a free y if $u_0' = u_0$. We denote by D' the resulting blue path on $v(D) + v(G_0')$ vertices. Since at most one red edge incident to its vertex was selected while connecting D and G_0', in view of (1) we have

$$\begin{aligned} |\mathrm{Red}(D')| &\le |\mathrm{Red}_G(G_2' \cup G_1')| + \frac{2}{5}v(L) + |\mathrm{Red}_G(G_0')| + 1 \\ &\le \left\lceil \frac{2}{5}v(G_1' \cup G_2') \right\rceil + \frac{2}{5}v(L) + \left\lceil \frac{2}{5}v(G_0') \right\rceil = \left\lceil \frac{2}{5}\mathrm{ess}(G) \right\rceil. \end{aligned}$$

In the last equality we used the property of a good graph that one of the numbers $v(G_1' \cup G_2')$, $v(G_0')$ is divisible by 5 and, clearly, $2v(L)/5$ is integer.

In all cases we obtain a blue path D' on $v(L) + v(G_0') + v(G_1') + v(G_2') + v(L) = \mathrm{ess}(G) = n$ vertices, such that $|\mathrm{Red}(D')| \le \lceil 2\mathrm{ess}(G)/5 \rceil = \lceil 2n/5 \rceil$. The third stage ends.

Observe that the set $\mathrm{Red}(D')$ for the path D' obtained at the end of Stage 3 is also the set of all red edges selected in the game. Indeed, G was good at the end of Stage 2 so every red edge was incident to $G_0' \cup G_1' \cup G_2' \cup L$ then. Furthermore every red edge selected during the third stage was incident to L or to the path D with $V(D) = V(L) \cup V(G_1') \cup V(G_2')$. It follows from the construction of D' that $D \subseteq D'$. It may have happened that $G_0' \not\subseteq D'$, but only if G_0' had one vertex. However, in such a case at the end of Stage 2 the number of red edges incident to G_0' was at most $\lceil 2v(G_0')/5 \rceil - 1 = 0$. Thus the number of red edges selected in the game is $|\mathrm{Red}(D')| \le \lceil 2n/5 \rceil$.

Notice also that at the end of the third stage there are no blue edges on the board other than the edges of D' so the number of blue edges selected in the game $\mathrm{RR}_\emptyset(P_4, P_n)$ is $n-1$.

Summarising, the number of all coloured edges (and thus the number of rounds of the game $\mathrm{RR}_\emptyset(P_4, P_n)$) is not greater than

$$n - 1 + \left\lceil \frac{2}{5}n \right\rceil = \left\lceil \frac{7}{5}n \right\rceil - 1.$$

This ends the proof in all cases apart from the exceptional case.

We proceed to the exceptional case. Let us recall that then we assume that $r = 2$, after the second stage the position on the board consists of a red path $P = ab$ and the coloured graph H (vertex-disjoint from P) which is very good, with essential subgraphs (G_0, G_1, G_2, L) and m essential vertices. The game proceeds to the third stage.

Stage 3 in the exceptional case.

We consider three subcases, depending on the essential subgraphs of H.

Subcase 1. $G_0 \neq \emptyset$.

Let u_0, u_0' be the ends of G_0. Builder selects edges u_0a and $u_0'b$. Since Painter avoids a red P_4, he has to colour blue at least one of these edges. If both are blue, we obtain a blue $(1,1)$ path B with 1-ends a, b, on $v(G_0) + v(P)$ vertices. If exactly one of the edges is blue, we get an extended $(2,0)$-path on $v(G_0) + 2$ vertices. In both cases, since $5 \mid v(G_0)$, the new coloured component $H' = B \cup P \cup \{u_0a, u_0'b\}$ on the board satisfies

$$|\mathrm{Red}(B)| \le |\mathrm{Red}_H(G_0)| + |\mathrm{Red}(P)| \le \frac{2}{5}v(G_0) - 1 + 2 = \frac{2}{5}v(G_0) + \left\lceil \frac{2}{5}v(P) \right\rceil = \left\lceil \frac{2}{5}v(B) \right\rceil.$$

Thus H' is a simple good graph. Since $H \setminus G_0$ is very good, the assumption of Lemma 4.2 is satisfied by $H \setminus G_0$ and H'. Based on this lemma, after a few further rounds the host graph is a good graph G with

$$\mathrm{ess}(G) = \mathrm{ess}(H \setminus G_0) + \mathrm{ess}(H') = \mathrm{ess}(H) + 2 = m + r = n.$$

Further analysis is the same as in Stage 3 for non-exceptional cases above. It leads to the conclusion that a blue path of length $n-1$ is created and the number of edges on the board at this moment is not greater than $\lceil 7n/5\rceil - 1$.

Subcase 2. $G_0 = \emptyset$ and $G_1 \neq \emptyset$.

Let u_1, u_1' be the 1-ends of G_1. Builder forces blue edges u_1a and $u_1'b$. As a result we obtain a blue $(1,1)$-path G_1' with 1-ends a and b, on $v(G_1)+v(P)$ vertices. Let $G = H\cup P\cup \{u_1a, u_1'b\}$. Since $5 \mid v(G_1\cup G_2)$, the subgraph $G_1'\cup G_2$ of the coloured graph G satisfies

$$\begin{aligned} |\mathrm{Red}_G(G_1'\cup G_2)| &\leq |\mathrm{Red}_H(G_1\cup G_2)| + |\mathrm{Red}_G(P)| \leq \frac{2}{5}v(G_1\cup G_2)+1 = \frac{2}{5}v(G_1\cup G_2) + \left\lceil \frac{2}{5}v(P)\right\rceil \\ &= \left\lceil \frac{2}{5}(v(G_1\cup G_2)+v(P))\right\rceil = \left\lceil \frac{2}{5}(v(G_1'\cup G_2))\right\rceil. \end{aligned}$$

It is not hard to verify that G satisfies all conditions of a good graph and its essential subgraphs are $(\emptyset, G_1', G_2, L)$. Again we use the same argument as in Stage 3 for non-exceptional cases and conclude the assertion.

Subcase 3. $G_0 = \emptyset$ and $G_1 = \emptyset$.

Then $G_2 \neq \emptyset$ since G has at least 10 essential vertices. Let w_2 be the transition vertex of G_2 and u_2 be its red end. Let us recall that the edge w_2u_2 is red. Builder forces blue edges: w_2a, au_2 and u_2b. Thereby we obtain a blue $(1,0)$-path F on $v(G_2)+v(P)$ vertices and its 1-end b.

Apart from the extended $(2,0)$-path G_2', H may contain also a limb. Then, as in Stage 3 for non-exceptional case, Builder transforms the limb into a blue $(1,0)$-path P' with an 1-end x_4 and $|\mathrm{Red}(P')| \leq 2 = \frac{2}{5}v(L)$. Afterwards, Builder forces the edge x_4b blue and a longer blue path B is obtained.

Let $B = F\cup P'\cup\{x_4a\}$ if $L\neq\emptyset$, while otherwise we put $B = F$ and define $P' = \emptyset$. Note that $|\mathrm{Red}(P')| \leq \frac{2}{5}v(L)$ in both cases ($L=\emptyset$ or not). Notice also that in both cases $V(B) = V(G_2\cup L\cup P)$ so $v(B) = \mathrm{ess}(H)+2 = n+r = n$. So there is a blue path on n vertices on the board and Builder wins.

Furthermore, since $5 \mid v(G_2\cup G_1) = v(G_2)$, we have

$$\begin{aligned} |\mathrm{Red}(B)| &\leq |\mathrm{Red}_H(G_2)| + |\mathrm{Red}(P)| + |\mathrm{Red}(P')| \leq \frac{2}{5}v(G_2)+1+\frac{2}{5}v(L) \\ &= \frac{2}{5}v(G_2) + \left\lceil\frac{2}{5}v(P)\right\rceil + \frac{2}{5}v(L) = \left\lceil \frac{2}{5}v(B)\right\rceil. \end{aligned}$$

It follows from the strategy of Builder that $\mathrm{Red}(B)$ consists of all red edges of the game $\mathrm{RR}_\emptyset(P_4, P_n)$ and the blue path B contains all blue edges present on the board. Thus the number of rounds in the game is

$$e(B) + |\mathrm{Red}(B)| \leq \left\lceil\frac{7}{5}v(B)\right\rceil - 1 = \left\lceil \frac{7}{5}n\right\rceil - 1.$$

This ends the proof in the exceptional case and the proof of Theorem 1.4.

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